ANALYTIC AND NUMERICAL SOLUTIONS TO THE DELAY DIFFERENTIAL EQUATION $y'(t) = \alpha y(t - \delta)$, REVISED

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Abstract. This paper discusses the linear first Order Delay Differential Equation (DDE) $y'(t) = \alpha y(t - \delta)$, $\delta > 0$, on the interval $[0, b], b > 0$ and some of its applications. A Proof is given that for any arbitrary "pre-function" on $[-\delta, 0]$, the solution is unique. We obtain an infinite set of characteristic solutions and analyze various cases. The results show how the growth rate constant, α, and the delay constant, δ, are related to the existence of oscillatory solutions. The method is extended to cover the equation $y'(t) = \alpha_1 y(t-\delta)+\alpha_2 y(t)$, and a computer algortihm for obtaining the numerical solution is provided.

Genesis

This paper was first presented by me at the Joint Northern and Southern California Sections of the Mathematics Association of America in San Luis Obispo, California, July 1995. I put a copy of it on my Sonoma State University web page and it has subsequently been cited by other authors writing papers on delay differential equations.

1. Introduction

We present both analytic and numerical solutions to the Boundary Value Problem (BVP)

$$(1.1) \qquad \begin{cases} y'(t) = \alpha y(t - \delta), \delta > 0, \text{ on } [0, b], \ b > 0 \\ y(t) = \phi(t), \text{ on } [-\delta, 0], \end{cases}$$

where $\phi \in \mathcal{C}^1[-\delta, 0]$, and α, and δ, are any real numbers, with $\delta > 0$.

The differential equation in this problem is a Delay Differential Equation(DDE); the interval $[-\delta, 0]$ is called the "pre-interval" and the function ϕ is called the "pre-function." The differential equation in the BVP is a first order DDE and has, as expected, solutions exhibiting exponential growth or decay. What is, perhaps, *unexpected* is that this same first order equation also has solutions that are oscillatory. The nature of each such solution depends upon the relationship between α and δ. Of course, the pre-function ϕ can also influence the solution.

DDEs are special cases of *functional differential equations* in which the derivative of the unknown function, y has a value at t, that is equal to a function of y at some *other* function of t.

For example, the equation

$$y'(t) = f(t, y(t), y[u(t)]) \tag{1.2}$$

is the general first order functional differential equation.

This type of equation is useful in modeling various complex phenomena in which the change in a quantity is dependent, not only upon its state at a given time, but also upon how that quantity is affected, over time, by some mechanism. The change in room temperature controlled by a thermostat may come to mind. These are extremely difficult equations to solve analytically, except for certain special cases, and as mentioned before, the solutions often have surprising properties.

When the function $u(t)$ in Equation (1.2) is given in terms of an earlier value of t i.e. $(u(t) < t)$ then (1.2) is called a *delay differential equation.* Some examples are: $u(t) = pt$, where $0 < p < 1$, or $u(t) = t - q$, where $q > 0$.

A specific example is *logistic age-structured population growth:* $x'(t) = k(L - x(t-\delta))x(t)$. Here the change in population is jointly proportional to the current population, $x(t)$, and the "room to grow" population, $L - x(t - \delta)$ at a prior time.

2. Uniqueness

We will prove that the solution to the Boundary Value Problem (1) is unique by showing that if $\phi(t) \equiv 0$ on $[-\delta, 0]$, then the problem has only the trivial solution on $[0, b]$. This will permit us to verify that we are finding *the* solution to (1.1) when we use the method of characteristics.

Theorem 1.

$$\begin{cases} y'(t) = \alpha y(t-\delta), \delta > 0 \ \textit{on} \ [0, b], b > 0 \\ y(t) = 0 \ \textit{on} \ [-\delta, 0] \end{cases} \tag{2.1}$$

has $y(t) \equiv 0$ *on* $[-\delta, b]$ *as its only solution.*

Note: If $b > \delta$*, our proof will show that* $y \equiv 0$ *is the solution on the interval* $[0, \delta]$*, then if* $b > 2\delta$*, we shift the differential equation to the interval* $[\delta, 2\delta]$*, making* $[0, \delta]$ *the new pre interval, on which* $y = 0$ *will serve as the new pre-function which will let us solve the problem only on* $[0, 2\delta]$*. Otherwise, if* $\delta < b < 2\delta$*, we have extended our solution to the interval* $[0, b]$*. Continuing this way we will be able to move the* *solution along to cover the interval* $[0, b]$ *for any positive real number* b.

Proof. First, the function, $y \equiv 0$ on $[0, \delta]$, is a solution as can be seen by direct substitution. Next, we note that the differential equation itself, (without the boundary condition) is *linear*. Thus, if $x(t)$ and $y(t)$ are any two solutions then $x'(t) = \alpha x(t-\delta)$ and $y'(t) = \alpha y(t-\delta)$. Also, if we define a function $w(t) = C_1 x(t) + C_2 y(t)$ for any two constants C_1, C_2, then $w'(t) = \alpha w(t-\delta)$. This means that $w(t)$ is also a solution to the differential equation.

Given that $y(t) \equiv 0$ is one solution, let us suppose, for contradiction, there exists another function $x(t)$, *not identically zero* that also satisfies (2.1). Thus, the function $x(t)$ satisfies the differential equation on $[0, \delta]$ and the pre-function, 0, on $[-\delta, 0]$, but takes on a nonzero value at least once somewhere in the half open interval $(0, \delta]$. That is, we are assuming that $x(\tau) \neq 0$ for some number $\tau \in (0, \delta]$.

Define M to be the set of reals such that $s \in M$ iff either $s = -\delta$ or $s > -\delta$ and $x(t) = 0 \ \forall t \in [-\delta, s]$. The set M exists since it contains all of the points in the interval $[-\delta, 0]$. Furthermore, M is bounded above, since τ is one of its upper bounds.

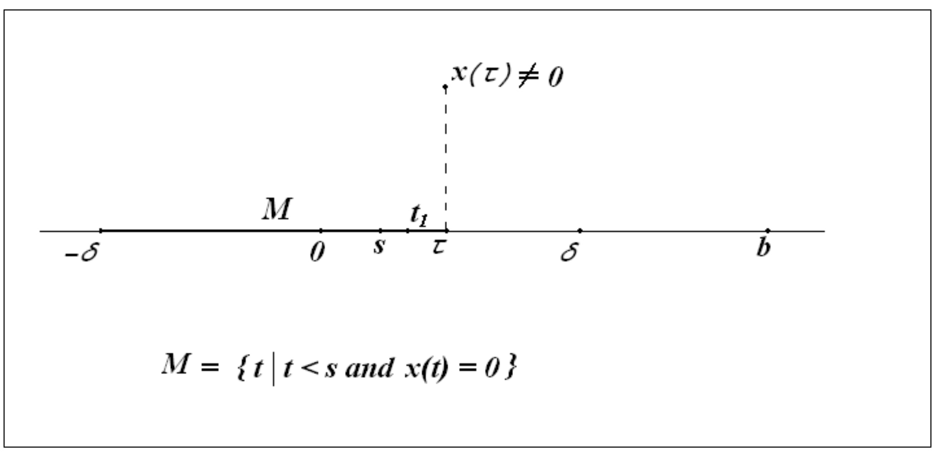

Figure 1 the set, M.

Let t_1 be the Least Upper Bound (LUB) of M. Note that $x(t_1) = 0$, otherwise $\exists$ a positive number, ε such that $x(t) \neq 0$ on the open interval $(t_1 - \varepsilon, t_1 + \varepsilon)$, making $t_1 - \varepsilon$ an upper bound of M, less than the LUB of M. Let $t_2 = t_1 + \frac{\delta}{2}$, then there exists a number t_0, between t_1 and t_2 such that $x(t_0) \neq 0$. If there is not any t_0 for which this is true then $x(t) = 0$, for all t between t_1 and t_2 making t_1 not an upper bound of M. Since x is continuous then there exists an interval $[p, r]$ containing t_0 as an interior point and such that $\forall t \in [p, r], x(t) \neq 0$. Let q be the minimum of r and t_2. Therefore $x(t) \neq 0$ on the interval $[p, q]$ where $q \leq t_2$.

Now let N be the number set such that $s \in N$ iff either $s = q$ or $s < q$ and $x(t) \neq 0 \forall t \in (s, q]$. We know that N exists since t_0 is an element of it. Since $x(t_1) = 0$, N is bounded below because t_1 is one of its lower bounds. Let θ be the Greatest Lower Bound (GLB) of N. Observe that since x is continuous at θ then $x(\theta) = 0$, otherwise there would be a positive number ϵ such that x would be nonzero throughout the segment $(\theta - \epsilon, \theta + \epsilon)$, making θ not a lower bound of N. Denote N by $(\theta, q]$.

Since for all $t \in N$, $t < t_2 = t_1 + \frac{\delta}{2}$, then $t - \delta \in M$ and $x(t - \delta) = 0$. So, from the differential equation $x'(t) = \alpha x(t-\delta)M$. Hence, $x'(t) \equiv 0$ on $(\theta, q]$. This means that $x(t) =$ a constant, C on $(\theta, q]$. But $x(\theta) = 0$, so by continuity of x at θ, the constant must be zero. Therefore $x(t) \equiv 0$ on $(\theta, q]$ contradicting the assumption that $x(t_0) \neq 0$ at some point in $[t_1, b]$. □

Theorem 2. *If each of $x(t)$ and $y(t)$ is a solution to the BVP (1.1), then $x(t) \equiv y(t)$ on $[-\delta, b]$.*

Proof. Let $w(t) = x(t) - y(t)$, then

$$\begin{aligned} w'(t) &= x'(t) - y'(t) \\ &= \alpha x(t - \delta) - a y(t - \delta) \\ &= \alpha w(t - \delta) \text{ on } (0, b]. \end{aligned}$$

Also, on $[-\delta, 0]$, $x(t) = y(t) = \phi(t)$; so $w(t) = 0$. Therefore $w(t)$ is the trivial solution satisfying equation (2.1); thus, $x(t) \equiv y(t)$ on $[-\delta, b]$. □

3. Characteristic Equations

Let us attempt to solve (1.1) by the method of characteristics. Following Hale[2], assume a solution $y(t) = Ce^{rt}$, substitute into $y'(t) = \alpha t(t - \delta), \delta \neq 0$ and we get the nonlinear characteristic equation $re^{r\delta} - \alpha = 0$. Take δ as a fixed positive number and define the function $f(r)$ as

$$f(r) = re^{r\delta} - \alpha. \tag{3.1}$$

In Figure 2, we sketch a few members of this one-parameter family of curves. Here α is the parameter.

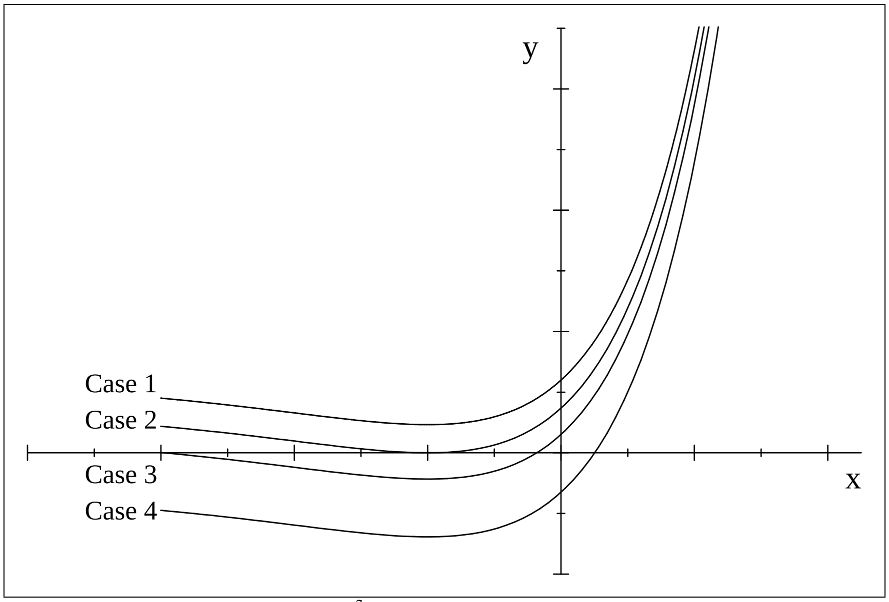

Figure 2. $f(r) = re^{\delta r} - \alpha$ for a fixed δ and various α.

What we want are the complex roots of $f(r) = 0$, that is

$$re^{r\delta} - \alpha = 0. \tag{3.2}$$

First, let us dispose of the case in which $\alpha = 0$. In this case, the DDE becomes $y'(t) = 0$ and equation (3.2) has $r = 0$ as its only root. Hence the solution is the constant $\phi(0)$.

It is clear from the graphs in Figure 2 that when $\alpha \neq 0$, we get the following four cases. In some of these there are real solutions, but it turns out that in all cases, this equation has infinitely many complex (non-real) solutions. With $\delta > 0$, consider the intercept $-\alpha$. We characterize roots of $f(r)$ as falling into one of the following four possibilities:

Case 1: If $\alpha < -\frac{1}{\delta e} < 0$, then $f(r)$ has no real zeros,
Case 2: If $\alpha = -\frac{1}{\delta e}$, then $f(r)$ has exactly one real zero, $r = -\frac{1}{\delta}$,
Case 3: If $-\frac{1}{\delta e} < \alpha < 0$, then $f(r)$ has exactly 2 real roots, both negative, and
Case 4: If $\alpha > 0$, then $f(r)$ has exactly one real root, r, and $r > 0$.

4. The Characteristic Solutions

4.1. **Case 1.** $\alpha < -\frac{1}{\delta e} < 0.$. Here $f(r)$ has no real roots, but we can seek complex numbers $z = p \pm iq$ such that $ze^{z\delta} - \alpha = 0$. If

$$\begin{aligned} (p+iq)e^{\delta(p+iq)} - \alpha &= 0, \text{then} \\ (p+iq)e^{iq\delta} &= \alpha e^{-p\delta}, \text{ or} \\ (p+iq)(\cos(q\delta) + i\sin(q\delta)) &= \alpha e^{-p\delta}. \end{aligned}$$

Multiplying these out, we get the following equations for the real and imaginary parts.

$$p\cos(q\delta) - q\sin(q\delta) = \alpha e^{-p\delta}, \text{ and} \tag{4.1}$$

$$q\cos(q\delta) + p\sin(q\delta) = 0, \text{ or} \tag{4.2}$$

$$p = -q\cot(q\delta), \ q \neq 0. \tag{4.3}$$

Notice that

$$\lim_{q\to 0} -q\cot(\delta q) = \lim_{q\to 0} \frac{-q\delta\cos(q\delta)}{\delta\sin(q\delta)} = \frac{-1}{\delta}.$$

Therefore $p \to -\frac{1}{\delta}$ as $q \to 0$. Continuity at $q = 0$ implies that Equations (4.1) and (4.2) are satisfied by $(p,q) = (\frac{-1}{\delta}, 0)$. Hence we get $\alpha = \frac{-1}{\delta e}$, when $q = 0$. This produces the single root $r = \frac{-1}{\delta}$ of Case 2.

Continuing Case 1 with $q \neq 0$, we substitute p from Equation (4.3) into (4.1) getting

$$q = -\alpha\sin(q\delta)e^{q\delta\cot(q\delta)}. \tag{4.4}$$

Let $X = q\delta$, then we may rewrite (4.4) as

$$X = -\delta\alpha\sin(X)e^{X\cot(X)}, \text{ where } -\delta\alpha > \frac{1}{e}. \tag{4.5}$$

We solve (4.5) by finding the intersections of the line $Y = X$, with the one-parameter family of curves

$$Y = -\delta\alpha\sin(X)e^{X\cot(X)}. \tag{4.6}$$

Here, δ is fixed and α is the parameter.

The graph in Figure 3 is for the case in which $\alpha < \frac{-1}{\delta e} < 0$, and it shows that Equation (4.5) has infinitely many solutions; denote them by X_k, $k = 1, 2, 3, ...$. We can use Newton's Method to obtain solutions for various given values of α. Specific examples will be considered later in this discussion.

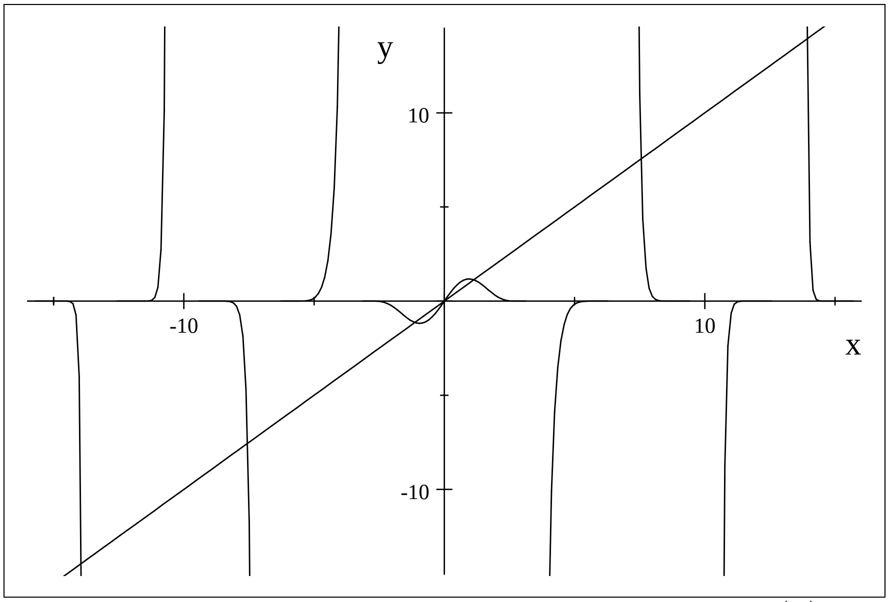

Figure 3. Graphs of $Y = X$ and $Y = -\delta\alpha\sin(X)e^{X\cot(X)}$.

Since $q = X/\delta$, we can define $q_k = X_k/\delta$. From Equation (4.3) we get p_k. Thus, the roots of Equation (4.5) are $p_k + iq_k$, and the characteristic solutions are $e^{p_k t}\cos(q_k t)$ and $e^{p_k t}\sin(q_k t)$. Since the BVP (1.1) is linear and $\alpha < -\frac{1}{\delta e}$, in this case, the formal solution to the DDE is

$$y(t) = \sum_{k=1}^{\infty} e^{p_k t}(C_{1k}\cos(q_k t) + C_{2k}\sin(q_k t)) \tag{4.7}$$

where C_{1k} and C_{2k} are arbitrary constants.

In this case, $q_k \neq 0$, otherwise $p_k = -1/\delta$ making $\alpha = -1/\delta e$, contrary to $\alpha < -1/\delta e$. An important thing to notice about the point (X, Y) is that, when $X > 0$, the family of curves defined by Equation (4.6) are intersected to the right of the vertical asymptotes that are odd multiples of π. The values of p_k are non-positive at all these points of intersection and the values decrease as $|X| \to \infty$.

If for some positive integers m and k, $\alpha = -(4m+1)\pi/2\delta = q_k$, then $p_k = 0$ for that k: so, the solutions are oscillatory and undamped. But $p_k < 0$ for all other values of q_k, and the oscillations in (4.7) are damped by the amplitude $e^{p_k t}$.

Let us consider a special case in which $p_k = 0$ giving us undamped oscillations. Choose $C_{1k} = 1$, $C_{2k} = C_{1j} = C_{2j} = 0$ for all $j \neq k$, then $q_k = -(4k+1)\pi/2\delta = \alpha$, and (4.7) is $y(t) = \cos(\alpha t)$, satisfying $y'(t) = \alpha y(t-\delta)$. A particular example would be that if $\alpha = -\frac{9\pi}{2\delta}$, then $y = \cos(\frac{9\pi}{2\delta}t)$ is a solution to $y'(t) = -\frac{9\pi}{2\delta}y(t-\delta)$, which is easy to check.

4.2. **Case 2.** If $\alpha = -\frac{1}{\delta e}$ then $f(r)$ has one real root $r = -\frac{1}{\delta}$ in addition to the complex roots that can be found by Equations (4.1) and (4.2). The real root, $p = -\frac{1}{\delta}$ when $q = 0$, gives us the additional characteristic solution $e^{(-1/\delta)t}$. Now the solution to the BVP (1.1) is

$$y(t) = C_0 e^{(-1/\delta)t} + \sum_{k=1}^{\infty} e^{p_k t}(C_{1k}\cos(q_k t) + C_{2k}\sin(q_k t)), \tag{4.8}$$

where p_k and q_k are roots of (4.1) and (4.2) for this α.

4.3. **Case 3.** If $-\frac{1}{\delta e} < \alpha < 0$, then $f(r) = re^{r\delta} - \alpha$ has two real roots, both negative. One of them is to the right of $-\frac{1}{\delta}$, and the other is to the left. To solve for either one use Newton's Method. For example if r_0 is the root greater than $-\frac{1}{\delta}$, start with $\rho_0 = -\frac{1}{2\delta}$ and for each positive integer k, define ρ_{k+1} as $\rho_k - f(\rho_k)/f'(\rho_k)$. Then $r_0 = \lim_{k\to\infty}\rho_k$. Similarly to find the root less than $-\frac{1}{\delta}$, define the sequence with an initial value of $\rho_0 = -\frac{2}{\delta}$. Call this root, r_1. The two additional characteristic solutions $e^{r_1 t}$, and $e^{r_2 t}$ combined with the characteristic solutions obtained from Equations (4.1) and (4.2) gives us the following solution to Equation (1.1) when $\alpha \in (-\frac{1}{\delta e}, 0)$

$$y(t) = C_1 e^{r_0 t} + C_2 e^{r_1 t} + \sum_{k=1}^{\infty} e^{p_k t}(C_{1k}\cos(q_k t) + C_{2k}\sin(q_k t)). \tag{4.9}$$

4.4. **Case 4.** If $\alpha > 0$, the only real root of $re^{r\delta} - \alpha = 0$ is positive. Denote this root by r and use Newton's method starting with $\rho_0 = 1$ to get it. So when $\alpha > 0$, the solution to (1.1) is

$$y(t) = C_3 e^{rt} + \sum_{k=1}^{\infty} e^{p_k t}(C_{1k}\cos(q_k t) + C_{2k}\sin(q_k t)). \tag{4.10}$$

5. The General Solution

In all four cases $p_k \le 0$, and $p_k = 0$ was disposed of as discussed in Case 1. Taking $p_k < 0$, all the infinite series solutions in each of the Equations (4.7) through (4.10) are convergent. We may summarize the formal characteristic solution in all cases as follows.

Theorem 3. *Let δ be any positive number, α any real nonzero number and $p_k \pm iq_k$ complex roots to (3.2) obtained from (4.1) and (4.2), then for arbitrary constants C_{1k} and C_{2k} the function $y(t)$ defined as follows*

(5.1)
$$y(t) = C_0 e^{-t/\delta} + C_1 e^{r_0 t} + C_2 e^{r_1 t} + C_3 e^{rt} + \sum_{k=1}^{\infty} e^{p_k t}(C_{1k}\cos(q_k t) + C_{2k}\sin(q_k t))$$

satisfies the equation

$$y'(t) = \alpha y(t-\delta) on [0, b], b > 0,$$

provided that

(i) $C_0 = C_1 = C_2 = C_3 = 0$ when $\alpha < -\frac{1}{\delta e}$,
(ii) $C_1 = C_2 = C_3 = 0$ and C_0 is arbitrary when $\alpha = -\frac{1}{\delta e}$,
(iii) $C_0 = C_3 = 0$ and $C1$ & C_2 are arbitrary and r_0 and r_1 are the real roots of Equation (3.2) when $-\frac{1}{\delta e} < \alpha < 0$.
(iv) $C_0 = C_1 = C_2 = 0$ and C_3 is arbitrary and r is the real root of Equation (3.2) when $\alpha > 0$.

Proof. The proof is outlined in the above discussion of the various cases. □

In order to solve the BVP in Equation (1.1), we would like to use Equation (5.1) for a given pair α, δ and a given function $\phi(t)$ and impose the condition that for $t \in [-\delta, 0]$,

$$\begin{aligned} \phi(t) \quad = \quad & \\ & C_0 e^{-t/\delta} + C_1 e^{r_0 t} + C_2 e^{r_1 t} + C_3 e^{rt} + \\ & \sum_{k=1}^{\infty} e^{p_k t}(C_{1k}\cos(q_k t) + C_{2k}\sin(q_k t)) \end{aligned} \tag{5.2}$$

We need to determine the arbitrary coefficients, with some of the first four coefficients already set = 0 according to the given relationship between α and δ. If we could do this then we would have the solution to (1.1) since it is unique as shown in Theorem 2.

6. Approximate Solutions

The characteristic functions $\{e^{p_k t}\cos(q_k t), e^{p_k t}\sin(q_k t)\}$ are cleraly linearly independent, Therefore, we may approximate the solution given in equation (5.1) by defining a function $y_n(t)$ as follows

$$
\begin{aligned}
y_n(t) \;&=\\
&C_0e^{-t/\delta}+C_1e^{r_0t}+C_2e^{r_1t}+C_3e^{rt}+\\
&\sum_{k=1}^{n}e^{p_kt}(C_{1k}\cos(q_kt)+C_{2k}\sin(q_kt)).
\end{aligned}
\tag{6.1}
$$

We could set $y_n(t)=\phi(0)$ for continuity at 0, then we can uniformly partition $[-\delta,0]$ into m subintervals where $m=2n-1+h$ points. Here h is either $0,1$, or 2 according to the number of arbitrary coefficients among the first four. If $\mathcal{P}_m$ is the aforementioned partition and its points are: $-\delta=t_0<t_1<\ldots<t_m=0$, then for $j=0,1,\ldots,m-1$, the equation $y_n(t_j)=\phi(t_j)$, is an m by m nonsingular linear system that can be solved for its coefficients. Thus, we have an approximate solution based upon the partition $\mathcal{P}_m$. In Section 7 below, we give an approximate solution $y_2(t)$ for Case 1. The we will follow this with comparison to a numerical solution in Section 8.

7. A particular example illustrating Case 1

As a particular example, we take $\alpha=-1.25,\delta=1.25$, and $\phi=e^{-t^2}$ on $[-\delta,0]$. Let us solve Equation (1.1) on the interval [0,30]. Here, the given values of α and δ put us into Case 1 and Equation (4.5) becomes $X=1.5625sin(X)e^{X\cot(X)}$, (here $X=q\delta$). Solving for a few roots by Newton's Method, we get $X=\pm1.5684$, ±7.6465, ±13.9808,.... From these we get values of $q:q=1.2547$, 6.1172, 11.1847. Now, from Equation (4.3), $p=-0.0031$, -1.2877, -1.7629. Our approximate solution is $y_2(t)$, with $y_2(0)=\phi(0)$, and sampling ϕ on the interval $[-\delta,0]$ at points $t_1=-0.41666$, $t_2=-0.8333$, and $t_3=-1.25$. By requiring that $y_2(t)=\phi(t)$ at these three points, we get the system

$$
\begin{aligned}
e^{t^2} \;&=\; e^{-.0031t}(C_{11}\cos(1.2547t)+C_{12}\sin(1.2547t)+\\
&\quad e^{-1.2877t}(C_{21}\cos(6.1172t)+C_{22}\sin(6.1172t))
\end{aligned}
$$

at the four points, $t=0$, t_1, t_2, t_3.

Solving for the coefficients, we get the following approximation to the characteristic solution:

$$
\begin{aligned}
y_2(t) \;&=\; e^{-0.0031t}(.9666\cos(1.2547t)-.0510\sin(1.2547t))\\
&\quad +e^{-1.2877t}(.03334\cos(6.1172t)-.02488\sin(6.1172t)
\end{aligned}
\tag{7.1}
$$

If we plot this solution we get the graph in Figure 4.

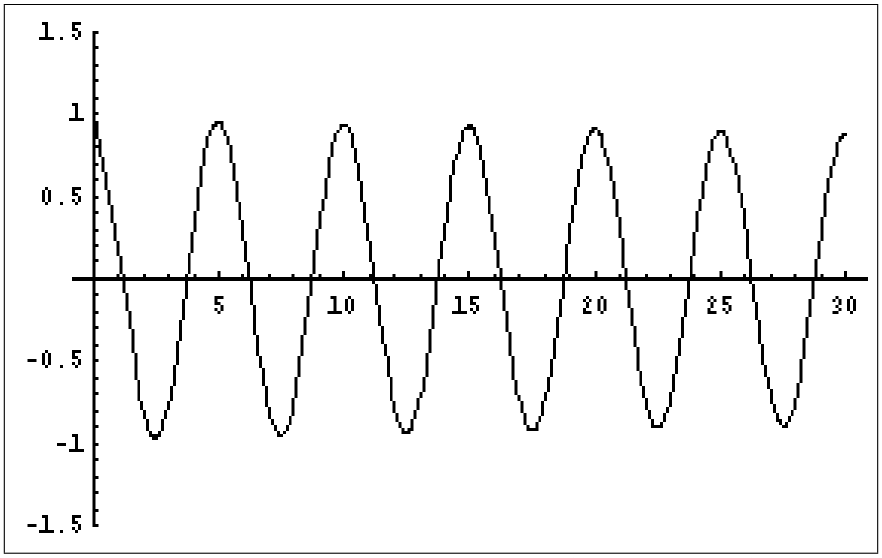

Figure 4. Approximate solution (Equation 7.1)

8. Numerical Solution (Pseudocode)

We are going to apply a numerical algorithm that treats the BVP(1) as a finite difference equation. In particular, we will use *Euler's Method.* A higher order numerical method such as the fourth order Runge-Kutta is not any better here since the differential equation is linear. Here is a pseudocode for the BVP Equation (1.1).

(1) User input B, the right-end-point of the interval $[0, b]$.
(2) User input the coefficient ALPHA and delay size DELTA, or α and δ.
(3) User input N, the number of subdivisions of the interval $[0, b]$.
(4) User input function subroutine PHI(T) for the prefunction $\phi(t)$ on $[0, b]$.
(5) Declare arrays Y(N) and T(N).
(6) Compute H=B/N, the step size fo the interval $[0, b]$.
(7) Compute K=DELTA/N, the number of steps in the preinterval $[-\delta, 0]$
(8) Compute M=N+K, the total number of steps in $[-\delta, b]$.
(9) Set graphics mode and parameters.
(10) Prefunction output
    (a) Loop1
        (i) I=0 to I=K
        (ii) T(I)= - DELTA+I*H
        (iii) Y(I)=PHI(T(I))
        (iv) Plot the point (T(I)),Y(I)).

(b) End Loop1

(11) Solution output

(a) Loop2

(i) I=K to I=M

(ii) T(I)=(I-K)*H.

(iii) Y(I)=Y(I-1)+ALPHA*H*Y(I-K).

(iv) Plot the point (T(I),Y(I)).

(b) End Loop2

(12) END

Figure 5 is the graph of the numerical solution obtained from a *Mathematica* implementation of the pseudocode given here.

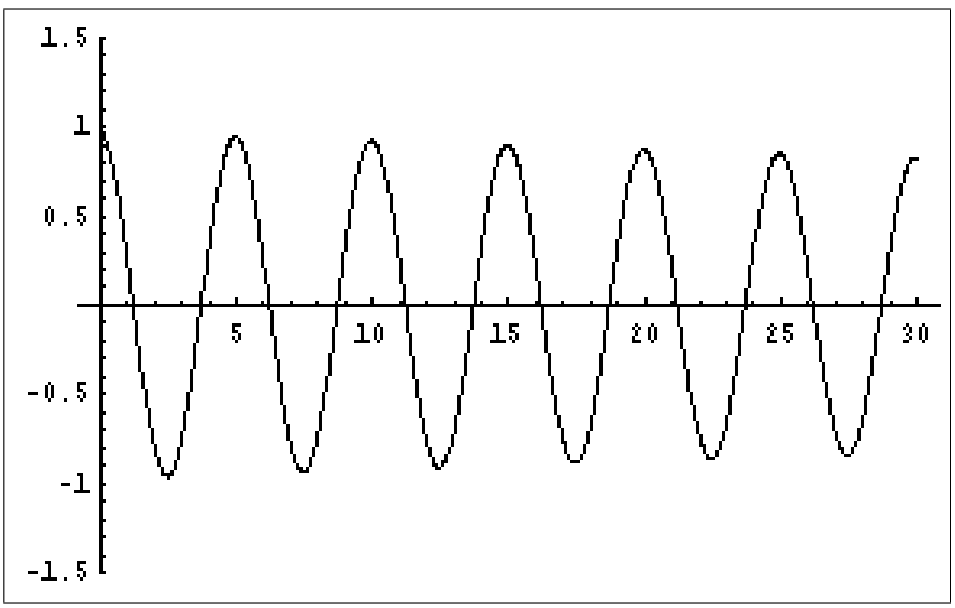

Figure 5. Numerical Solution

If we superimpose the two graphs, Figure 4 and Figure 5, we will get the graph in Figure 6, comparing the approximate solution obtained from Equation (7.1) and the numerical solution.

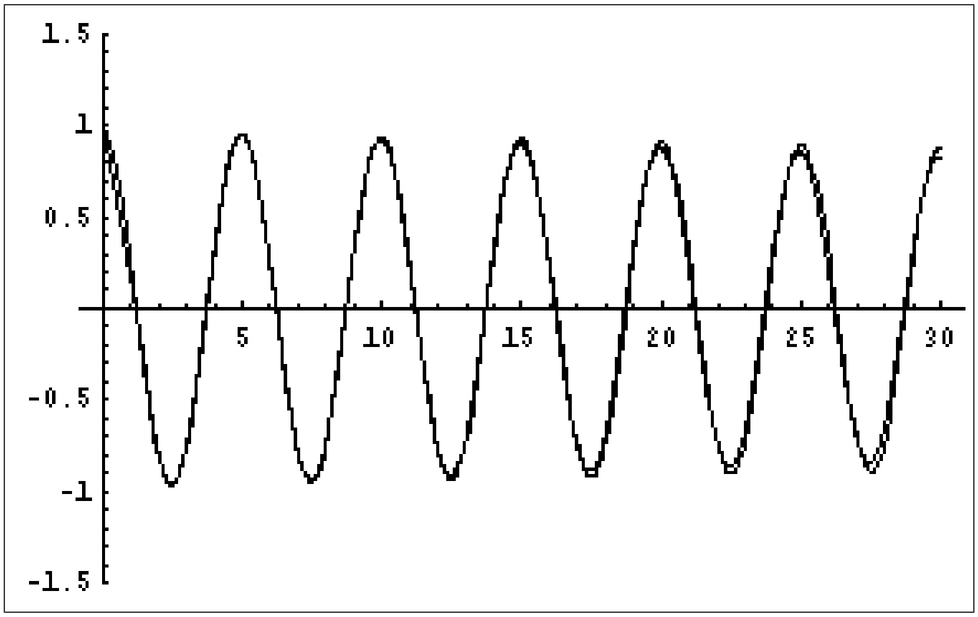

Figure 6 Overlay of the two solutions.

9. Extension to the equation $y'(t) = \alpha y(t-\delta) + \beta y(t)$

Given $\delta > 0$, we can apply the characteristic method to the boundary value problem

$$\begin{cases} y'(t) = \alpha y(t-\delta) + \beta y(t) \ , \delta > 0, \text{ on } [0,b], b > 0 \\ y(t) = \phi(t), \text{ on } [-\delta, 0] \end{cases} \tag{9.1}$$

as follows. Let $y(t) = e^{mt}$, then from Equation (9.1), we get

$$(m-\beta)e^{m\delta} - \alpha = 0. \tag{9.2}$$

Now, letting $m - \beta = s$, this becomes

$$se^{s\delta} - \alpha e^{-\beta\delta} = 0.$$

Since α, β, and δ are given, we can write $\alpha e^{-\beta\delta}$ as a single number, α_1, getting

$$se^{s\delta} - \alpha_1 = 0. \tag{9.3}$$

Now we solve Equation (9.3) for s in the same way as we solved Equation (3.2) for r.

10. References

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