A Tale of Two Problems With the Same Differential Equation

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Let a and b be real numbers. We wish to solve and compare the following two deceptively simple looking problems.

Problem 1. Find the function $x(t) \in \mathcal{C}^1(-\infty, \infty)$, such that

$$\begin{aligned} x'(t) &= ax(-t) \text{ for all } t \quad (1) \\ x(0) &= b \end{aligned}$$

Problem 2. Find the function $x(t) \in \mathcal{C}^0(-\infty, \infty)$, such that

$$\begin{aligned} x'(t) &= ax(-t) \text{ for all } t > 0 \quad (2) \\ x(t) &= b \text{ for all } t \leq 0 \end{aligned}$$

These both have the same differential equation and the same value at $t = 0$. We may be surprised to learn that their solutions are radically different from one another. Each problem has a unique solution, but the solution to (1) is an oscillating, periodic function with period $\frac{2\pi}{a}$,on $(-\infty, \infty)$, while the solution to (2) is the union of two straight rays, a horizontal one on $(-\infty, 0]$ and one with slope ab on $(0, \infty)$.

The solution to Problem 1 is

$$x(t) = b\cos(at) + b\sin(at) \quad (3)$$

The solution to Problem 2 is

$$\begin{aligned} x(t) &= abt + b, \text{for all } t > 0 \\ x(t) &= b, \text{for all } t \leq 0 \quad (4) \end{aligned}$$

For specific choices of $(a, b) = (1, 1)$ the solutions are depicted in Figure 1

This puzzle is resolved by noticing that the differential equation is a Functional Differential Equation (FDE) and is not completely determined by the equation and an initial condition alone. In particular, Problem 2 requires that we be supplied with sufficient information regarding the behavior of $x(t)$ on the interval $(-\infty, 0]$. Here, we arbitrarily chose $x(t)$ to be the constant b; any other function $p(t)$, with $p(0) = b$, would give us a different solution to Problem 2, without affecting the solution to Problem 1. Let us now state a generalization of Problem 2, show how to solve it, and then show how to solve Problem 1.

Generalization of Problem 2.

Let a and b be real numbers and $p(t) \in C^1(-\infty, 0]$, such that $p(0) = b$. Find a function $x(t) \in \mathcal{C}^0(-\infty, \infty)$ such that

$$\begin{aligned} x'(t) &= ax(-t) \text{ for all } t > 0 \quad (5) \\ x(t) &= p(t) \text{ for all } t \leq 0 \end{aligned}$$

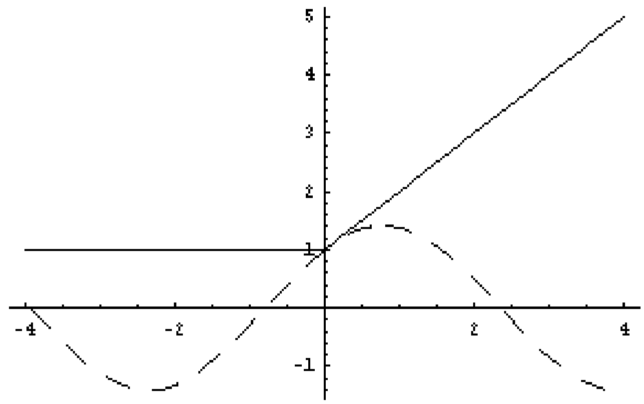

Figure 1: Figure 1

Solution:

When $t > 0$, then $-t < 0$ so the right-hand-side of (5), $ax(-t)$, becomes $ap(-t)$. We simply integrate $ap(-t)$ and adjust the integration constant to make the solution continuous at 0.

For example, if $p(t) = b + t$, when $t \leq 0$, then solving $x'(t) = a(b - t)$ we get $x(t) = -a(b - t)^2/2 + c$. In order to make x continuous at 0,we choose $c = b + ab^2/2$. Thus when $p(t) = b + t$, the solution is:

$$\begin{aligned} x(t) &= b + abt - at^2/2 \text{ for } t > 0 \qquad (6) \\ x(t) &= b + t \text{ for } t \leq 0 \end{aligned}$$

How do we solve Problem 1? Differentiate Equation (1), obtaining $x''(t) = -ax'(-t)$, then replace $x'(-t)$ by $ax(-(-t))$. Thus, we get

$$\begin{aligned} x''(t) &= -a^2x(t) \text{ for all } t \qquad (7) \\ x(0) &= b \\ x'(0) &= ab \end{aligned}$$

This is a second order ordinary differential equation and its only solution is the only solution to Equation (1).